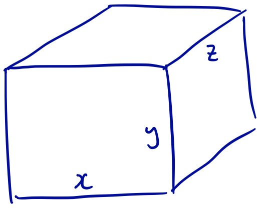
Example 3. A rectangular box is to be made from 100 m^2 of cardboard. Find the maximum volume of such a box.



Let x, y, z be the lengths of the box sides.

We want to

maximize $\underbrace{xyz}_{f(x,y,z)}$

subject to $\underbrace{\cancel{2}xy + \cancel{2}xz + \cancel{2}yz}_{g(x,y,z)} = \cancel{100}\ 50$

$$\nabla f(x,y,z) = \langle yz,\ xz,\ xy \rangle$$

$$\nabla g(x,y,z) = \langle y+z,\ x+z,\ x+y \rangle$$

LM equations:

$$yz = \lambda(y+z) \quad ①$$
$$xz = \lambda(x+z) \quad ②$$
$$xy = \lambda(x+y) \quad ③$$
$$xy + xz + yz = 50 \quad ④$$

Note: if $\lambda = 0$

①, ②, ③ $\Rightarrow yz + xz + xy = 0$

which contradicts ④!

$\Rightarrow \lambda$ must be $\neq 0$

x① $\Rightarrow xyz = \lambda(xy + xz)$ ④

y② $\Rightarrow xyz = \lambda(xy + yz)$ ⑤

z③ $\Rightarrow xyz = \lambda(xz + yz)$ ⑥

④,⑤ $\Rightarrow \cancel{xy} + xz = \cancel{xy} + yz$

$\Rightarrow xz = yz \Rightarrow z = 0$ (7a) or $x = y$ (7b)

④,⑥ $\Rightarrow xy + \cancel{xz} = \cancel{xz} + yz$

$\Rightarrow xy = yz \Rightarrow y = 0$ (8a) or $x = z$ (8b)

⑤,⑥ $\Rightarrow xy + \cancel{yz} = xz + \cancel{yz}$

$\Rightarrow xy = xz. \Rightarrow x = 0$ (9a) or $y = z$ (9b)

7a / 8a / 9a $\Rightarrow$ $x=0,\ y=0,\ z=0$ $\Rightarrow$ contradicts ④ $\Rightarrow$ no solution

7a / 8b / 9b $\Rightarrow$ $z=0,\ \begin{array}{l} x=z=0 \\ y=z=0 \end{array}$ $\Rightarrow$ contradicts ④ $\Rightarrow$ no solution

7b / 8a / 9b and 7b / 8b / 9a are similar

7a / 8a / 9b $\Rightarrow$ $z=0,\ y=0$ ④ $\Rightarrow$ $0x + 0x + 0 = 50$ $\Rightarrow$ no solution

7a / 8b / 9a and 7b / 8a / 9a are similar

7b / 8b / 9b $\Rightarrow$ $x=y=z$ ④ $\Rightarrow$ $x^2 + x^2 + x^2 = 50$

$\Rightarrow$ $x = y = z = \sqrt{\frac{50}{3}}$

LM solutions: $\left(\sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}\right)$

$\Rightarrow f\left(\sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}, \sqrt{\frac{50}{3}}\right) = \left(\sqrt{\frac{50}{3}}\right)^3$ is an absolute maximum.

We know this because there exists x, y, z that satisfy the constraint w/ larger f values, e.g. $f(\underbrace{1, 1, 24.5}) = 24.5 < \left(\sqrt{\frac{50}{3}}\right)^3$.

$xy + yz + xz$
$= 1 + 24.5 + 24.5$
$= 50$